

Alternative Proof of the Product Rule

Theorem. Suppose f and g are differentiable at x . Then fg is differentiable at x , and

$$(fg)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

Proof.

$$\begin{aligned}(fg)'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left(f(x+h) \left(\frac{g(x+h) - g(x)}{h} \right) + g(x) \left(\frac{f(x+h) - f(x)}{h} \right) \right) \\ &= \lim_{h \rightarrow 0} f(x+h) \cdot \lim_{h \rightarrow 0} \left(\frac{g(x+h) - g(x)}{h} \right) + \lim_{h \rightarrow 0} g(x) \cdot \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} f(x+h) \cdot g'(x) + \lim_{h \rightarrow 0} g(x) \cdot f'(x) \\ &= f(x)g'(x) + g(x)f'(x). \quad \square\end{aligned}$$

Note that we have used the assumptions that $f'(x)$ and $g'(x)$ exist in the 2nd and 3rd to last equalities. Also, $\lim_{h \rightarrow 0} f(x+h) = f(x)$ because f is continuous at x , which follows from the assumption that f is differentiable at x .